

Geometric approach to the local Jacquet-Langlands correspondence

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ABSTRACT. In this paper, we give a purely geometric approach to the local Jacquet-Langlands correspondence for $\mathrm{GL}(n)$ over a p -adic field, under the assumption that the invariant of the division algebra is $1/n$. We use the ℓ -adic étale cohomology of the Drinfeld tower to construct the correspondence at the level of the Grothendieck groups with rational coefficients. Moreover, assuming that n is prime, we prove that this correspondence preserves irreducible representations. This gives a purely local proof of the local Jacquet-Langlands correspondence in this case. We need neither a global automorphic technique nor detailed classification of supercuspidal representations of $\mathrm{GL}(n)$.

1 Introduction

Let F be a p -adic field, i.e., a finite extension of \mathbb{Q}_p . Let $n \geq 1$ be an integer and D a central division algebra over F such that $\dim_F D = n^2$. The famous local Jacquet-Langlands correspondence gives a natural bijective correspondence between irreducible discrete series representations of $\mathrm{GL}_n(F)$ and irreducible smooth representations of D^\times . Let us recall its precise statement. Write $\mathbf{Irr}(D^\times)$ for the set of isomorphism classes of irreducible smooth representations of D^\times . We denote by $\mathbf{Disc}(\mathrm{GL}_n(F))$ the set of isomorphism classes of irreducible discrete series representations of $\mathrm{GL}_n(F)$. For $\rho \in \mathbf{Irr}(D^\times)$ (resp. $\pi \in \mathbf{Disc}(\mathrm{GL}_n(F))$), we denote the character of ρ (resp. π) by θ_ρ (resp. θ_π). Here θ_ρ is a locally constant function on D^\times , and θ_π is a locally integrable function on $\mathrm{GL}_n(F)$ which is locally constant on $\mathrm{GL}_n(F)^{\mathrm{reg}}$, the set of regular elements of $\mathrm{GL}_n(F)$. The precise statement of the local Jacquet-Langlands correspondence is the following:

Theorem 1.1 (the local Jacquet-Langlands correspondence) *There exists a unique bijection*

$$JL: \mathbf{Irr}(D^\times) \xrightarrow{\cong} \mathbf{Disc}(\mathrm{GL}_n(F))$$

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satisfying the following character relation: for every regular element h of D^\times , $\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h)$, where g_h is an arbitrary element of $\mathrm{GL}_n(F)$ whose minimal polynomial is the same as that of h .

The original proof of this theorem, due to Deligne-Kazhdan-Vigneras [DKV84] and Rogawski [Rog83], was accomplished by using a global automorphic method. In some cases, more explicit local studies can be found in [Hen93], [BH00], [BH05], which are based on the theory of types. However, apart from the case of $\mathrm{GL}(2)$, a purely local proof of Theorem 1.1 seems not to be known yet (*cf.* [Hen06, a comment after Theorem 2]).

In this article, under the assumption that the invariant of D is $1/n$, we will give a geometric approach to construct the bijection JL above. Let $R(D^\times)$ be the Grothendieck group of finite-dimensional smooth representations of D^\times , and $\overline{R}(\mathrm{GL}_n(F))$ the Grothendieck group of finite length smooth representations of $\mathrm{GL}_n(F)$ “modulo induced representations” (for a precise definition, see [Kaz86]). It is known that the classes of elements of $\mathbf{Irr}(D^\times)$ (resp. $\mathbf{Disc}(\mathrm{GL}_n(F))$) form a basis of $R(D^\times)$ (resp. $\overline{R}(\mathrm{GL}_n(F))$). Put $R(D^\times)_\mathbb{Q} = R(D^\times) \otimes_\mathbb{Z} \mathbb{Q}$ and $\overline{R}(\mathrm{GL}_n(F))_\mathbb{Q} = \overline{R}(\mathrm{GL}_n(F)) \otimes_\mathbb{Z} \mathbb{Q}$. The main theorems of this article are the following:

Theorem 1.2 (Theorem 6.6) *We can construct the following two homomorphisms geometrically:*

$$JL: R(D^\times)_\mathbb{Q} \longrightarrow \overline{R}(\mathrm{GL}_n(F))_\mathbb{Q}, \quad LJ: \overline{R}(\mathrm{GL}_n(F))_\mathbb{Q} \longrightarrow R(D^\times)_\mathbb{Q}.$$

These two maps are inverse to each other, and satisfy the character relations

$$\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h), \quad \theta_\pi(g_h) = (-1)^{n-1} \theta_{LJ(\pi)}(h)$$

for every regular $h \in D^\times$.

Theorem 1.3 (Theorem 6.10) *If n is prime, then JL induces a bijection*

$$JL: \mathbf{Irr}(D^\times) \xrightarrow{\cong} \mathbf{Disc}(\mathrm{GL}_n(F)).$$

Theorem 1.3 provides a purely local proof of Theorem 1.1 in the case above. In particular, the local Jacquet-Langlands correspondence for $\mathrm{GL}_2(F)$ and $\mathrm{GL}_3(F)$ are fully recovered.

The geometric object we use is the Drinfeld tower for $\mathrm{GL}_n(F)$. It is a tower of rigid spaces over (a disjoint union of) the $(n-1)$ -dimensional Drinfeld upper half space $\mathbb{P}_F^{n-1} \setminus \bigcup_H H$, where H runs through hyperplanes of \mathbb{P}_F^{n-1} defined over F (for more detailed explanation, see Section 2). Thanks to extensive studies by many people (*cf.* [Har97], [HT01], [Boy09], [Dat07]), it is now well-known that the local Jacquet-Langlands correspondence is realized in the ℓ -adic cohomology H_{Dr} of the Drinfeld tower. The methods in the works cited above are again global and automorphic. However, there is also a purely local study of the cohomology due

to Faltings [Fal94]. He began with an irreducible smooth representation ρ of D^\times , and investigated the ρ -isotypic part $H_{\text{Dr}}[\rho]$ of H_{Dr} by means of the Lefschetz trace formula. By this method, he succeeded to observe that the character relation in Theorem 1.1 appears naturally in $H_{\text{Dr}}[\rho]$. By using this result, we can give the map JL in Theorem 1.2.

To construct the inverse map LJ , we need to consider the opposite direction; we begin with an irreducible discrete series representation π of $\text{GL}_n(F)$ and investigate the “ π -isotypic part” of H_{Dr} . More precisely, we should consider the alternating sum of the extension groups $\sum_j (-1)^j \text{Ext}^j(H_{\text{Dr}}, \pi)$, since π is neither projective nor injective in general. To study it, we apply the method introduced in [Mie11]; namely, we use local harmonic analysis, such as transfer of orbital integrals. Furthermore, to prove Theorem 1.3, the non-cuspidality result obtained in [Mie10b] plays a crucial role.

Since our approach is entirely geometric, it is natural to expect that a similar argument may give interesting consequences for the mod- ℓ Jacquet-Langlands correspondence (*cf.* [Dat11]). The author also expects that our strategy can be extended to other Rapoport-Zink spaces, especially the Rapoport-Zink space for $\text{GSp}(4)$. He hopes to deal with these problems in future works.

We sketch the outline of this paper. In Section 2, we recall the definition of the Drinfeld tower and results in [Fal94]. We use the framework of [Dat00] to deal with finitely generated representations systematically. In Section 3, we study the alternating sum of the extension groups $\sum_j (-1)^j \text{Ext}^j(H_{\text{Dr}}, \pi)$ by means of local harmonic analysis. In Section 4, we apply another deep result of Faltings on the comparison of the Lubin-Tate tower and the Drinfeld tower. It provides a very important finiteness result on H_{Dr} . Under this finiteness, results in Section 2 and Section 3 can be written in a very simple form. After short preliminaries on representation theory in Section 5, finally in Section 6, we construct the maps JL and LJ in Theorem 1.2 by using the ℓ -adic cohomology of the Drinfeld tower, and investigate their properties.

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Notation For a totally disconnected locally compact group H , let $\mathbf{Irr}(H)$ be the set of isomorphism classes of irreducible smooth representations of H . We denote the Grothendieck group of finitely generated (resp. finite length) smooth H -representations by $K(H)$ (resp. $R(H)$). Put $R(H)_{\mathbb{Q}} = R(H) \otimes_{\mathbb{Z}} \mathbb{Q}$. For a finite-dimensional smooth representation σ of H , write θ_σ for the character of σ . It is a locally constant function on H . If moreover a Haar measure on H is fixed, we denote by $\mathcal{H}(H)$ the Hecke algebra of H , namely, the abelian group of locally constant compactly supported functions on H with convolution product. Put $\overline{\mathcal{H}}(H) = \mathcal{H}(H)/[\mathcal{H}(H), \mathcal{H}(H)] = \mathcal{H}(H)_H$ (the H -coinvariant quotient).

Let F be a p -adic field and \mathcal{O} its ring of integers. We denote the normalized valuation of F by v_F and the cardinality of the residue field of \mathcal{O} by q . Fix a uniformizer ϖ of \mathcal{O} . Denote the completion of the maximal unramified extension of \mathcal{O} by $\tilde{\mathcal{O}}$ and the fraction field of $\tilde{\mathcal{O}}$ by \tilde{F} .

Throughout this paper, we fix an integer $n \geq 1$. Let D be the central division algebra over F with invariant $1/n$, and \mathcal{O}_D its maximal order. Fix a uniformizer $\Pi \in \mathcal{O}_D$ such that $\Pi^n = \varpi$.

For simplicity, put $G = \mathrm{GL}_n(F)$. We denote by G^{reg} (resp. G^{ell}) the set of regular (resp. regular elliptic) elements of G . Write Z_G for the center of G . We apply these notations to other groups. For example, we write $(D^\times)^{\mathrm{reg}}$ for the set of regular elements of D^\times . As in Theorem 1.1, for $h \in (D^\times)^{\mathrm{reg}}$, let g_h be an element of G^{ell} whose minimal polynomial is the same as that of h . Such an element always exists, and is unique up to conjugacy. Moreover, $h \mapsto g_h$ induces a bijection between conjugacy classes in $(D^\times)^{\mathrm{reg}}$ and those in G^{ell} . Therefore, to $g \in G^{\mathrm{ell}}$ we can attach an element $h_g \in (D^\times)^{\mathrm{reg}}$ whose minimal polynomial is the same as that of g .

For a smooth G -representation π of finite length, we denote by θ_π the distribution character of π . It is a locally integrable function on G which is locally constant on G^{reg} .

We identify F^\times with Z_G and Z_{D^\times} . Then, we can consider the quotient groups $G/\varpi^\mathbb{Z}$ and $D^\times/\varpi^\mathbb{Z}$ under a discrete subgroup $\varpi^\mathbb{Z}$ of F^\times . We regard $\mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$ (resp. $\mathbf{Irr}(G/\varpi^\mathbb{Z})$) as a subset of $\mathbf{Irr}(D^\times)$ (resp. $\mathbf{Irr}(G)$). Similarly, $R(D^\times/\varpi^\mathbb{Z})$ (resp. $R(G/\varpi^\mathbb{Z})$) is regarded as a submodule of $R(D^\times)$ (resp. $R(G)$). Write $\mathbf{Disc}(G)$ for the set of isomorphism classes of irreducible discrete series representations of G . Put $\mathbf{Disc}(G/\varpi^\mathbb{Z}) = \mathbf{Disc}(G) \cap \mathbf{Irr}(G/\varpi^\mathbb{Z})$.

Fix Haar measures on G and D^\times . We endow $\varpi^\mathbb{Z}$ with the counting measure and consider the quotient measures on $G/\varpi^\mathbb{Z}$ and $D^\times/\varpi^\mathbb{Z}$. For $\varphi \in \mathcal{H}(G/\varpi^\mathbb{Z})$ and $g \in G^{\mathrm{ell}}$, put $O_g^{G/\varpi^\mathbb{Z}}(\varphi) = \int_{G/\varpi^\mathbb{Z}} \varphi(x^{-1}gx)dx$ (the orbital integral). It is well-known that this integral converges. Similarly, for $\varphi' \in \mathcal{H}(D^\times/\varpi^\mathbb{Z})$ and $h \in D^\times$, put $O_h^{D^\times/\varpi^\mathbb{Z}}(\varphi') = \int_{D^\times/\varpi^\mathbb{Z}} \varphi'(y^{-1}hy)dy$.

For a field k , we denote its algebraic closure by \bar{k} . Let ℓ be a prime which is invertible in \mathcal{O} . We fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and identify them. Every representation is considered over \mathbb{C} .

2 Drinfeld tower

Let us briefly recall the definition of the Drinfeld tower. For more detailed description, see [Dri76], [BC91], [RZ96, Chapter 3].

First of all, fix a special formal \mathcal{O}_D -module \mathbb{X} of \mathcal{O}_F -height n^2 over $\overline{\mathbb{F}}_q$. It is well-known that such \mathbb{X} is unique up to \mathcal{O}_D -isogeny.

We denote by \mathbf{Nilp} the category of $\tilde{\mathcal{O}}$ -schemes on which ϖ is locally nilpotent. Consider the functor $\mathcal{M}_{\mathrm{Dr}}$ from \mathbf{Nilp} to the category of sets that maps S to the set of isomorphism classes of pairs (X, ρ) consisting of

- a special formal \mathcal{O}_D -module X over S ,
- and an \mathcal{O}_D -quasi-isogeny $\rho: \mathbb{X} \otimes_{\overline{\mathbb{F}}_q} \overline{S} \longrightarrow X \otimes_S \overline{S}$,

where we put $\overline{S} = S \otimes_{\check{\mathcal{O}}} \overline{\mathbb{F}}_q$. Then \mathcal{M}_{Dr} is represented by a formal scheme locally of finite type over $\check{\mathcal{O}}$. We denote the formal scheme by \mathcal{M}_{Dr} again, and the rigid generic fiber of \mathcal{M}_{Dr} by M_{Dr} . It is known that M_{Dr} is the disjoint union of countable copies of the $(n-1)$ -dimensional Drinfeld upper half space.

For an integer $m \geq 0$, let $M_{\text{Dr},m}$ be the rigid space classifying Π^m -level structures on the universal formal \mathcal{O}_D -module over M_{Dr} . It is a finite étale Galois covering of M_{Dr} with Galois group $(\mathcal{O}_D/(\Pi^m))^\times$. The projective system $\{M_{\text{Dr},m}\}_{m \geq 0}$ is called the Drinfeld tower. We can define a natural right action of $G = \text{GL}_n(F)$ on each $M_{\text{Dr},m}$ because G is isomorphic to the group of self \mathcal{O}_D -quasi-isogenies of \mathbb{X} . On the other hand, D^\times also acts naturally on $M_{\text{Dr},m}$ on the right, since $1 + \Pi^m \mathcal{O}_D$ is a normal subgroup of D^\times .

Since M_m is too large (it has infinitely many connected components), we take the quotient $M_{\text{Dr},m}/\varpi^\mathbb{Z}$ of $M_{\text{Dr},m}$ by $\varpi^\mathbb{Z} \subset F^\times = Z_G \subset G$. Put

$$H_{\text{Dr},m}^i = H_c^i((M_{\text{Dr},m}/\varpi^\mathbb{Z}) \otimes_{\check{F}} \overline{F}, \overline{\mathbb{Q}}_\ell), \quad H_{\text{Dr}}^i = \varinjlim_m H_{\text{Dr},m}^i.$$

These are smooth representations of $G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}$. Unless $n-1 \leq i \leq 2(n-1)$, $H_{\text{Dr},m}^i = H_{\text{Dr}}^i = 0$.

Proposition 2.1 *The representation $H_{\text{Dr},m}^i$ is finitely generated as a G -module. Moreover, there exist compact open subgroups K_1, \dots, K_N of $G/\varpi^\mathbb{Z}$, $\varepsilon_\nu \in \{\pm 1\}$ and finite-dimensional smooth representations $\sigma_{m,\nu}$ of $K_\nu \times D^\times/\varpi^\mathbb{Z}$ for each $1 \leq \nu \leq N$ such that the following holds:*

$$\sum_i (-1)^i [H_{\text{Dr},m}^i] = \sum_{\nu=1}^N \varepsilon_\nu [\text{c-Ind}_{K_\nu}^{G/\varpi^\mathbb{Z}} \sigma_{m,\nu}] \quad \text{in } K(G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}).$$

Proof. We only give a sketch of a proof, since it seems to be well-known (cf. [Far04, Proposition 4.4.13]).

It is known that $M_{\text{Dr}}/\varpi^\mathbb{Z}$ has an open covering $\mathfrak{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ consisting of quasi-compact open subsets, indexed by the set Λ of vertices of the Bruhat-Tits building of $\text{PGL}_n(F)$. This covering satisfies the following properties:

- (a) For $g \in G/\varpi^\mathbb{Z}$, $U_\lambda \cdot g = U_{g^{-1}\lambda}$.
- (b) For each $\lambda \in \Lambda$, there exist only finitely many $\lambda' \in \Lambda$ satisfying $U_\lambda \cap U_{\lambda'} \neq \emptyset$.
- (c) For each $\lambda \in \Lambda$, $K_\lambda = \{g \in G/\varpi^\mathbb{Z} \mid U_\lambda \cdot g = U_\lambda\}$ is a compact open subgroup of $G/\varpi^\mathbb{Z}$.

For a finite subset $I \subset \Lambda$, put

$$U_I = \bigcap_{\lambda \in I} U_\lambda, \quad K_I = \{g \in G/\varpi^\mathbb{Z} \mid U_I \cdot g = U_I\}.$$

For an integer $r \geq 0$, set $\Lambda_r = \{I \subset \Lambda \mid \#I = r+1, U_I \neq \emptyset\}$. Then, by the three properties above, we have the following:

- For each $r \geq 0$, the set of $G/\varpi^{\mathbb{Z}}$ -orbits in Λ_r is finite.
- We have $\Lambda_r = \emptyset$ for sufficiently large r .
- For every finite subset $I \subset \Lambda$, K_I is a compact open subgroup of $G/\varpi^{\mathbb{Z}}$.

Take a system of representatives $I_{r,1}, \dots, I_{r,N_r}$ of $(G/\varpi^{\mathbb{Z}}) \backslash \Lambda_r$ and put $K_{r,i} = K_{I_{r,i}}$.

Let $\mathfrak{U}_m = \{U_{m,\lambda}\}_{\lambda \in \Lambda}$ be the covering obtained as the inverse image of \mathfrak{U} . For a finite subset $I \subset \Lambda$, put $U_{m,I} = \bigcap_{\lambda \in I} U_{m,\lambda}$. Then we have the Čech spectral sequence

$$E_1^{-r,s} = \bigoplus_{I \in \Lambda_r} H_c^s(U_{m,I} \otimes_{\check{F}} \check{F}, \overline{\mathbb{Q}}_\ell) \implies H_{\text{Dr},m}^{-r+s}.$$

Put

$$V_{m,r,i}^s = H_c^s(U_{m,I_{r,i}} \otimes_{\check{F}} \check{F}, \overline{\mathbb{Q}}_\ell).$$

It is a finite-dimensional smooth representation of $K_{r,i} \times D^\times / \varpi^{\mathbb{Z}}$ and vanishes for $s > 2n$ (cf. [Hub96, Proposition 5.5.1, Proposition 6.3.2]). We can easily observe that $E_1^{-r,s}$ is isomorphic to $\bigoplus_{i=1}^{N_r} \text{c-Ind}_{K_{r,i}}^{G/\varpi^{\mathbb{Z}}} V_{m,r,i}^s$ as a $G/\varpi^{\mathbb{Z}} \times D^\times / \varpi^{\mathbb{Z}}$ -representation. Therefore $E_1^{-r,s}$ is a finitely generated $G/\varpi^{\mathbb{Z}}$ -representation, and vanishes for all but finitely many (r, s) . Hence $H_{\text{Dr},m}^i$ is finitely generated as a G -module (cf. [Ber84, Remarque 3.12]). Moreover, in $K(G/\varpi^{\mathbb{Z}} \times D^\times / \varpi^{\mathbb{Z}})$ we have

$$\sum_i (-1)^i [H_{\text{Dr},m}^i] = \sum_{r,s} (-1)^{-r+s} E_1^{-r,s} = \sum_{r,s} \sum_{i=1}^{N_r} (-1)^{-r+s} \text{c-Ind}_{K_{r,i}}^{G/\varpi^{\mathbb{Z}}} V_{m,r,i}^s.$$

This concludes the proof. ■

Definition 2.2 We denote by η_m the image of $\sum_i (-1)^i [H_{\text{Dr},m}^i]$ under the rank map

$$\text{Rk}: K(G/\varpi^{\mathbb{Z}} \times D^\times / \varpi^{\mathbb{Z}}) \longrightarrow \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}} \times D^\times / \varpi^{\mathbb{Z}})$$

(cf. [Dat00, 1.2]). For $h \in D^\times$, we define $\eta_{m,h} \in \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}})$ by

$$\eta_{m,h}(g) = \int_{D^\times / \varpi^{\mathbb{Z}}} \eta_m(g, h'^{-1} h h') dh'.$$

Using the expression of $\sum_i (-1)^i [H_{\text{Dr},m}^i]$ in Proposition 2.1, we can give more explicit description of η_m and $\eta_{m,h}$:

Proposition 2.3 For $m \geq 0$ and $h \in D^\times$, define $\tilde{\eta}_m \in \mathcal{H}(G/\varpi^{\mathbb{Z}} \times D^\times / \varpi^{\mathbb{Z}})$ and $\tilde{\eta}_{m,h} \in \mathcal{H}(G/\varpi^{\mathbb{Z}})$ by

$$\tilde{\eta}_m = \sum_{\nu=1}^N \frac{\varepsilon_\nu \theta_{\sigma_{m,\nu}^\vee}}{\text{vol}(K_\nu \times D^\times / \varpi^{\mathbb{Z}})}, \quad \tilde{\eta}_{m,h} = \sum_{\nu=1}^N \frac{\varepsilon_\nu \theta_{\sigma_{m,\nu}^\vee}(-, h)}{\text{vol}(K_\nu)},$$

where $(-)^{\vee}$ denotes the contragredient, and $\theta_{\sigma_{m,\nu}^\vee}$ (resp. $\theta_{\sigma_{m,\nu}^\vee}(-, h)$) is regarded as a function on $G/\varpi^{\mathbb{Z}} \times D^\times / \varpi^{\mathbb{Z}}$ (resp. $G/\varpi^{\mathbb{Z}}$) by setting $\theta_{\sigma_{m,\nu}^\vee}(g, h) = 0$ for $g \notin K_\nu$.

Then, the image of $\tilde{\eta}_m$ (resp. $\tilde{\eta}_{m,h}$) in $\overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}} \times D^\times / \varpi^{\mathbb{Z}})$ (resp. $\overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}})$) coincides with η_m (resp. $\eta_{m,h}$).

Proof. The assertion for $\tilde{\eta}_{m,h}$ immediately follows from that for $\tilde{\eta}_m$. Thus it suffices to prove the following:

Let K be a compact open subgroup of $G/\varpi^{\mathbb{Z}}$. For every finite-dimensional smooth representation σ of $K \times D^\times/\varpi^{\mathbb{Z}}$, the image of $\text{vol}(K)^{-1}\theta_{\sigma^\vee}$ in $\overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}} \times D^\times/\varpi^{\mathbb{Z}})$ coincides with $\text{Rk}([\text{c-Ind}_K^{G/\varpi^{\mathbb{Z}}} \sigma])$.

Since the image of $[\sigma]$ under the rank map $\text{Rk}: K(K \times D^\times/\varpi^{\mathbb{Z}}) \longrightarrow \overline{\mathcal{H}}(K \times D^\times/\varpi^{\mathbb{Z}})$ is $\text{vol}(K \times D^\times/\varpi^{\mathbb{Z}})^{-1}\theta_{\sigma^\vee}$, this claim follows from the commutative diagram below (cf. [Dat00, proof of Theorem 1.6]):

$$\begin{array}{ccc} K(K \times D^\times/\varpi^{\mathbb{Z}}) & \xrightarrow{\text{Rk}} & \overline{\mathcal{H}}(K \times D^\times/\varpi^{\mathbb{Z}}) \\ \text{c-Ind}_K^{G/\varpi^{\mathbb{Z}}} \downarrow & & \downarrow \text{extension by 0} \\ K(G/\varpi^{\mathbb{Z}} \times D^\times/\varpi^{\mathbb{Z}}) & \xrightarrow{\text{Rk}} & \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}} \times D^\times/\varpi^{\mathbb{Z}}). \end{array}$$

In [Fal94], Faltings investigated the function $\tilde{\eta}_m$ above by means of the Lefschetz trace formula. His results can be summarized in the following theorem.

Theorem 2.4 *Let $g \in G^{\text{reg}}$ and $h \in D^\times$.*

i) *If g is elliptic, then we have*

$$\begin{aligned} O_g^{G/\varpi^{\mathbb{Z}}}(\eta_{m,h}) &= \# \text{Fix}((g^{-1}, h^{-1}); M_{\text{Dr},m}/\varpi^{\mathbb{Z}}) \\ &= n \cdot \#\{a \in D^\times/\varpi^{\mathbb{Z}}(1 + \Pi^m \mathcal{O}_D) \mid hah_g^{-1} = a\}. \end{aligned}$$

ii) *If g is not elliptic, then we have $\int_{Z(g) \backslash G} \eta_{m,h}(x^{-1}gx)dx = 0$, where $Z(g)$ denotes the centralizer of g .*

Proof. Let us briefly recall the proof in [Fal94]. We use the notation in the proof of Proposition 2.1.

First consider the case where g is elliptic. Then, we can find a finite subset $\Lambda_g \subset \Lambda$ such that $g\Lambda_g = \Lambda_g$ and $gU_\lambda \cap U_\lambda = \emptyset$ for $\lambda \in \Lambda \setminus \Lambda_g$. Put $U_{m,g} = \bigcup_{\lambda \in \Lambda_g} U_{m,\lambda}$. Then $U_{m,g}$ is quasi-compact smooth and $(g^{-1}, h^{-1}): U_{m,g} \longrightarrow U_{m,g}$ has no fixed point on the boundary of $U_{m,g}$. Therefore we can apply the Lefschetz trace formula for this endomorphism (for a general theory of the Lefschetz trace formula for rigid spaces, see [Mie10a]). Noting that every fixed point of $(g^{-1}, h^{-1}): M_{\text{Dr},m}/\varpi^{\mathbb{Z}} \longrightarrow M_{\text{Dr},m}/\varpi^{\mathbb{Z}}$ lies in $U_{m,g}$, we obtain the following equality:

$$\sum_i (-1)^i \text{Tr}((g^{-1}, h^{-1}); H_c^i(U_{m,g} \otimes_{\check{F}} \overline{F}, \overline{\mathbb{Q}}_\ell)) = \# \text{Fix}((g^{-1}, h^{-1}); M_{\text{Dr},m}/\varpi^{\mathbb{Z}}).$$

By the Čech spectral sequence, we can easily show that the left hand side is equal to $O_g^{G/\varpi^{\mathbb{Z}}}(\tilde{\eta}_{m,h})$. The right hand side can be computed by using the period map, as in [Str08, §2.6]. The result is

$$\# \text{Fix}((g^{-1}, h^{-1}); M_{\text{Dr},m}/\varpi^{\mathbb{Z}}) = n \cdot \#\{a \in D^\times/\varpi^{\mathbb{Z}}(1 + \Pi^m \mathcal{O}_D) \mid hah_g^{-1} = a\}.$$

It is slightly different from [Fal94, Theorem 1], because our $M_{\text{Dr}}/\varpi^{\mathbb{Z}}$ is the disjoint union of n copies of Ω considered in [Fal94]. Rather, it is compatible with [Str08, Theorem 2.6.8]. This concludes the proof of i).

To prove ii), apply the same argument to $M_{\text{Dr},m}/\Gamma$, where Γ is a sufficiently small discrete torsion-free cocompact subgroup of $Z(g)$. ■

Corollary 2.5 *For $\rho \in \mathbf{Irr}(D^\times/\varpi^{\mathbb{Z}})$, $\text{Hom}_{D^\times}(\rho, H_{\text{Dr}}^i)$ is a finitely generated $G/\varpi^{\mathbb{Z}}$ -representation by [Mie11, Lemma 5.2]. The image of $\sum_i (-1)^i [\text{Hom}_{D^\times}(\rho, H_{\text{Dr}}^i)]$ under the map*

$$\text{Rk}^\vee : K(G/\varpi^{\mathbb{Z}}) \xrightarrow{\text{Rk}} \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}}) \xrightarrow{\vee} C^\infty(G^{\text{ell}})$$

coincides with $g \mapsto n\theta_\rho(h_g^{-1})$. Recall that for $f \in \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}})$ the locally constant function f^\vee on G^{ell} is given by $f^\vee(g) = \int_{G/\varpi^{\mathbb{Z}}} f(xg^{-1}x^{-1})dx$ (cf. [Dat00, p. 190]).

Proof. Take a sufficiently large integer $m \geq 0$ so that $\rho|_{1+\Pi^m\mathcal{O}_D}$ is trivial. Then we have $\text{Hom}_{D^\times}(\rho, H_{\text{Dr}}^i) = \text{Hom}_{D^\times}(\rho, (H_{\text{Dr}}^i)^{1+\Pi^m\mathcal{O}_D}) = \text{Hom}_{D^\times}(\rho, H_{\text{Dr},m}^i)$.

It is easy to see that the following diagram is commutative:

$$\begin{array}{ccc} K(G/\varpi^{\mathbb{Z}} \times D^\times/\varpi^{\mathbb{Z}}) & \xrightarrow{\text{Rk}} & \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}} \times D^\times/\varpi^{\mathbb{Z}}) \\ \text{Hom}_{D^\times}(\rho, -) \downarrow & & \downarrow (*) \\ K(G/\varpi^{\mathbb{Z}}) & \xrightarrow{\text{Rk}} & \overline{\mathcal{H}}(G/\varpi^{\mathbb{Z}}), \end{array}$$

where $(*)$ is given by

$$f \mapsto \left(g \mapsto \int_{D^\times/\varpi^{\mathbb{Z}}} f(g, h)\theta_\rho(h)dh \right).$$

Therefore, the image of $\sum_i (-1)^i [\text{Hom}_{D^\times}(\rho, H_{\text{Dr},m}^i)]$ under Rk^\vee can be calculated as follows:

$$\begin{aligned} g &\mapsto \int_{G/\varpi^{\mathbb{Z}}} \int_{D^\times/\varpi^{\mathbb{Z}}} \eta_m(xg^{-1}x^{-1}, h)\theta_\rho(h)dhdx \\ &= \frac{1}{\text{vol}(D^\times/\varpi^{\mathbb{Z}})} \int_{G/\varpi^{\mathbb{Z}}} \int_{D^\times/\varpi^{\mathbb{Z}}} \int_{D^\times/\varpi^{\mathbb{Z}}} \eta_m(xg^{-1}x^{-1}, h)\theta_\rho(h'h h'^{-1})dh'hdx \\ &= \frac{1}{\text{vol}(D^\times/\varpi^{\mathbb{Z}})} \int_{D^\times/\varpi^{\mathbb{Z}}} O_{g^{-1}}^{G/\varpi^{\mathbb{Z}}}(\eta_{m,h})\theta_\rho(h)dh \\ &= \frac{n}{\text{vol}(D^\times/\varpi^{\mathbb{Z}})} \int_{D^\times/\varpi^{\mathbb{Z}}} \#\{a \in D^\times/\varpi^{\mathbb{Z}}(1 + \Pi^m\mathcal{O}_D) \mid hah_g = a\}\theta_\rho(h)dh \\ &= \frac{n}{\#(D^\times/\varpi^{\mathbb{Z}}(1 + \Pi^m\mathcal{O}_D))} \\ &\quad \times \sum_{h \in D^\times/\varpi^{\mathbb{Z}}(1 + \Pi^m\mathcal{O}_D)} \#\{a \in D^\times/\varpi^{\mathbb{Z}}(1 + \Pi^m\mathcal{O}_D) \mid hah_g = a\}\theta_\rho(h) \end{aligned}$$

$$\begin{aligned}
 &= \frac{n}{\#(D^\times/\varpi^\mathbb{Z}(1 + \Pi^m \mathcal{O}_D))} \sum_{a \in D^\times/\varpi^\mathbb{Z}(1 + \Pi^m \mathcal{O}_D)} \theta_\rho(ah_g^{-1}a^{-1}) \\
 &= n\theta_\rho(h_g^{-1}).
 \end{aligned}$$

This completes the proof. ■

3 Some harmonic analysis

In this section, we use Theorem 2.4 to investigate the virtual D^\times -representation $\sum_{i,j \geq 0} (-1)^{i+j} \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)$. In the Lubin-Tate case, a similar study is carried out in [Mie11]. For $m \geq 0$, denote by K'_m the image of $1 + \Pi^m \mathcal{O}_D$ in $D^\times/\varpi^\mathbb{Z}$.

Lemma 3.1 *For a smooth representation V of $G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}$ and a smooth representation π of $G/\varpi^\mathbb{Z}$, we have $\text{Ext}_{G/\varpi^\mathbb{Z}}^j(V, \pi)^{K'_m} \cong \text{Ext}_{G/\varpi^\mathbb{Z}}^j(V^{K'_m}, \pi)$. Here $\text{Ext}_{G/\varpi^\mathbb{Z}}^j$ is taken in the category of smooth $G/\varpi^\mathbb{Z}$ -representations.*

Proof. First we prove that there exist a smooth $G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}$ -representation P which is projective as a $G/\varpi^\mathbb{Z}$ -representation and a $G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}$ -equivariant surjection $P \rightarrow V$. Take a $G/\varpi^\mathbb{Z}$ -equivariant surjection $P' \rightarrow V \rightarrow 0$ from a projective $G/\varpi^\mathbb{Z}$ -representation P' . Put $P = P' \otimes_{\mathbb{C}} C_c^\infty(D^\times/\varpi^\mathbb{Z})$. Then P is a smooth $G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}$ -representation which is projective as a $G/\varpi^\mathbb{Z}$ -representation, and a surjection $P \rightarrow V$ is naturally induced.

Therefore, we can take a $G/\varpi^\mathbb{Z} \times D^\times/\varpi^\mathbb{Z}$ -equivariant resolution $P_\bullet \rightarrow V \rightarrow 0$ of V such that P_i is projective as a smooth $G/\varpi^\mathbb{Z}$ -representation. Since $P_i^{K'_m}$ is a direct summand of P_i as a $G/\varpi^\mathbb{Z}$ -representation, $P_i^{K'_m}$ is also projective. Thus $P_\bullet^{K'_m} \rightarrow V^{K'_m} \rightarrow 0$ gives a projective resolution of $V^{K'_m}$. Hence we have

$$\begin{aligned}
 \text{Ext}_{G/\varpi^\mathbb{Z}}^j(V, \pi)^{K'_m} &= H^j(\text{Hom}_{G/\varpi^\mathbb{Z}}(P_\bullet, \pi))^{K'_m} \cong H^j(\text{Hom}_{G/\varpi^\mathbb{Z}}(P_\bullet^{K'_m}, \pi)) \\
 &= \text{Ext}_{G/\varpi^\mathbb{Z}}^j(V^{K'_m}, \pi),
 \end{aligned}$$

as desired. ■

Corollary 3.2 *For every $m \geq 0$ and $\pi \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$, we have*

$$\text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{K'_m} \cong \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr},m}^i, \pi).$$

It is finite-dimensional and vanishes if $j \geq n$. In particular, $\text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}$ is an admissible representation of $D^\times/\varpi^\mathbb{Z}$ and vanishes if $j \geq n$, where $(-)^{\text{sm}}$ denotes the set of $D^\times/\varpi^\mathbb{Z}$ -smooth vectors.

Proof. Lemma 3.1 tells us that

$$\text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{K'_m} = \text{Ext}_{G/\varpi^\mathbb{Z}}^j((H_{\text{Dr}}^i)^{K'_m}, \pi) = \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr},m}^i, \pi).$$

By Proposition 2.1 and [SS97, Corollary II.3.3], it is finite-dimensional and vanishes if $j \geq n$. ■

Remark 3.3 Later (Corollary 4.3) we will prove that $\text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr},m}^i, \pi)$ is in fact a finite-dimensional smooth D^\times -representation.

The character of $\sum_{i,j \geq 0} (-1)^{i+j} \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr},m}^i, \pi)$ can be computed by $\eta_{m,h}$ introduced in the previous section:

Proposition 3.4 For every $\pi \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$ and $h \in D^\times/\varpi^\mathbb{Z}$, we have

$$\sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(h; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr},m}^i, \pi)) = \text{Tr}(\eta_{m,h}; \pi) = \int_{G/\varpi^\mathbb{Z}} \eta_{m,h}(g) \theta_\pi(g) dg.$$

Proof. First note that $V \mapsto \sum_{j \geq 0} (-1)^j \text{Tr}(h, \text{Ext}_{G/\varpi^\mathbb{Z}}^j(V, \pi))$ induces a homomorphism $K(G/\varpi^\mathbb{Z}) \rightarrow \mathbb{C}$ of abelian groups. Therefore, by Proposition 2.1 and Proposition 2.3, we have only to show the following:

Let K be a compact open subgroup of $G/\varpi^\mathbb{Z}$. For every finite-dimensional smooth representation σ of $K \times D^\times/\varpi^\mathbb{Z}$ and $h \in D^\times$, we have

$$\sum_{j \geq 0} (-1)^j \text{Tr}(h; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(\text{c-Ind}_K^{G/\varpi^\mathbb{Z}} \sigma, \pi)) = \text{Tr}\left(\frac{\theta_{\sigma^\vee}(-, h)}{\text{vol}(K)}; \pi\right).$$

Since $\text{c-Ind}_K^{G/\varpi^\mathbb{Z}} \sigma$ is a projective $G/\varpi^\mathbb{Z}$ -representation, $\text{Ext}_{G/\varpi^\mathbb{Z}}^j(\text{c-Ind}_K^{G/\varpi^\mathbb{Z}} \sigma, \pi) = 0$ for $j \geq 1$. Take an open normal subgroup $K_1 \subset K$ such that $\sigma|_{K_1}$ is trivial. Then the left hand side can be computed as follows:

$$\begin{aligned} \sum_{j \geq 0} (-1)^j \text{Tr}(h; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(\text{c-Ind}_K^{G/\varpi^\mathbb{Z}} \sigma, \pi)) &= \text{Tr}(h; \text{Hom}_{G/\varpi^\mathbb{Z}}(\text{c-Ind}_K^{G/\varpi^\mathbb{Z}} \sigma, \pi)) \\ &= \text{Tr}(h; \text{Hom}_K(\sigma, \pi|_K)) = \text{Tr}(h; \text{Hom}_{K/K_1}(\sigma, \pi^{K_1})) \\ &= \frac{1}{\#(K/K_1)} \sum_{g \in K/K_1} \text{Tr}((g^{-1}, h^{-1}); \sigma) \text{Tr}(g; \pi^{K_1}) \\ &= \frac{1}{\text{vol}(K)} \text{Tr}(\theta_{\sigma^\vee}(-, h); \pi). \end{aligned}$$

This completes the proof. ■

Lemma 3.5 For every $g \in G^{\text{ell}}$, $h \in D^\times$ and an integer $m \geq 0$, we have

$$O_g^{G/\varpi^\mathbb{Z}}(\eta_{m,h}) = n O_{h_g}^{D^\times/\varpi^\mathbb{Z}}\left(\frac{\mathbf{1}_{hK'_m}}{\text{vol}(K'_m)}\right),$$

where $\mathbf{1}_{hK'_m}$ denotes the characteristic function of hK'_m .

Proof. By Theorem 2.4 i), we obtain

$$\begin{aligned} O_g^{G/\varpi^\mathbb{Z}}(\eta_{m,h}) &= n \cdot \#\{a \in (D^\times/\varpi^\mathbb{Z})/K'_m \mid hah_g^{-1} = a\} \\ &= \frac{n}{\text{vol}(K'_m)} \int_{D^\times/\varpi^\mathbb{Z}} \mathbf{1}_{hK'_m}(ah_g a^{-1}) da = n O_{h_g}^{D^\times/\varpi^\mathbb{Z}}\left(\frac{\mathbf{1}_{hK'_m}}{\text{vol}(K'_m)}\right). \end{aligned} \quad \blacksquare$$

Next recall the definition of a transfer of a test function.

Definition 3.6 For $\varphi \in \mathcal{H}(G)$ and $\varphi^D \in \mathcal{H}(D^\times)$, we say that φ^D is a transfer of φ if

$$\int_{D^\times/\varpi^\mathbb{Z}} \varphi^D(y^{-1}h_g y) dy = (-1)^{n-1} \int_{G/\varpi^\mathbb{Z}} \varphi(x^{-1}gx) dx$$

for every $g \in G^{\text{ell}}$.

We know that if $\varphi \in \mathcal{H}(G)$ is supported on G^{ell} , then it has a transfer $\varphi^D \in \mathcal{H}(D^\times)$ (cf. [Mie11, Lemma 3.2]). The following lemma is obvious:

Lemma 3.7 Assume that $\varphi \in \mathcal{H}(G)$ is supported on G^{ell} and let $\varphi^D \in \mathcal{H}(D^\times)$ be its transfer. Put

$$\varphi_\varpi(g) = \sum_{i \in \mathbb{Z}} \varphi(\varpi^i g), \quad \varphi_\varpi^D(h) = \sum_{i \in \mathbb{Z}} \varphi^D(\varpi^i h).$$

Then $\varphi_\varpi \in \mathcal{H}(G/\varpi^\mathbb{Z})$, $\varphi_\varpi^D \in \mathcal{H}(D^\times/\varpi^\mathbb{Z})$ and $O_{h_g}^{D^\times/\varpi^\mathbb{Z}}(\varphi_\varpi^D) = (-1)^{n-1} O_g^{G/\varpi^\mathbb{Z}}(\varphi_\varpi)$ for every $g \in G^{\text{ell}}$.

Theorem 3.8 Assume that $\varphi \in \mathcal{H}(G)$ is supported on G^{ell} and let $\varphi^D \in \mathcal{H}(D^\times)$ be its transfer. Then, for every $\pi \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$ we have

$$\sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi^D; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}) = (-1)^{n-1} n \text{Tr}(\varphi; \pi).$$

Proof. Let φ_ϖ and φ_ϖ^D be as in the previous lemma. Then clearly we have

$$\begin{aligned} \sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi^D; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}) &= \sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi_\varpi^D; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}), \\ (-1)^{n-1} n \text{Tr}(\varphi; \pi) &= (-1)^{n-1} n \text{Tr}(\varphi_\varpi; \pi). \end{aligned}$$

Thus we may replace φ and φ^D by φ_ϖ and φ_ϖ^D , respectively.

Take $m \geq 0$ such that φ_ϖ^D is K'_m -invariant, and write

$$\varphi_\varpi^D = \sum_{h \in J} a_h \frac{\mathbf{1}_{hK'_m}}{\text{vol}(K'_m)},$$

where J is a finite subset of $D^\times/\varpi^\mathbb{Z}$ and $a_h \in \mathbb{C}$. By Corollary 3.2 and Proposition 3.4, we have

$$\begin{aligned} \sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi_\varpi^D; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}) &= \sum_{i,j \geq 0, h \in J} (-1)^{i+j} a_h \text{Tr}(h; \text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr},m}^i, \pi)) \\ &= \sum_{h \in J} a_h \int_{G/\varpi^\mathbb{Z}} \eta_{m,h}(g) \theta_\pi(g) dg. \end{aligned}$$

By Theorem 2.4 ii) and Weyl's integral formula (cf. [Kaz86, Theorem F]), we have

$$\int_{G/\varpi^{\mathbb{Z}}} \eta_{m,h}(g) \theta_{\pi}(g) dg = \sum_T \frac{1}{\#W_T} \int_{T^{\text{reg}}/\varpi^{\mathbb{Z}}} D(t) O_t^{G/\varpi^{\mathbb{Z}}}(\eta_{m,h}) \theta_{\pi}(t) dt,$$

where T runs through conjugacy classes of elliptic maximal tori of G , W_T denotes the rational Weyl group of T and $D(t)$ denotes the Weyl denominator (cf. [Rog83, p. 185]). The measure dt on $T/\varpi^{\mathbb{Z}}$ is normalized so that the volume of $T/\varpi^{\mathbb{Z}}$ is one. Lemma 3.5 and Lemma 3.7 tell us that

$$\begin{aligned} \sum_{h \in J} a_h O_t^{G/\varpi^{\mathbb{Z}}}(\eta_{m,h}) &= \sum_{h \in J} n a_h O_{h_t}^{D^{\times}/\varpi^{\mathbb{Z}}} \left(\frac{\mathbf{1}_{hK'_m}}{\text{vol}(K'_m)} \right) = n O_{h_t}^{D^{\times}/\varpi^{\mathbb{Z}}}(\varphi_{\varpi}^D) \\ &= (-1)^{n-1} n O_t^{G/\varpi^{\mathbb{Z}}}(\varphi_{\varpi}) \end{aligned}$$

for every $t \in T^{\text{reg}}$. By Weyl's integral formula again, we have

$$\begin{aligned} \sum_{i,j \geq 0} (-1)^{i+j} \text{Tr}(\varphi_{\varpi}^D; \text{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\text{Dr}}^i, \pi)^{\text{sm}}) \\ = (-1)^{n-1} n \sum_T \frac{1}{\#W_T} \int_{T^{\text{reg}}/\varpi^{\mathbb{Z}}} D(t) O_t^{G/\varpi^{\mathbb{Z}}}(\varphi_{\varpi}) \theta_{\pi}(t) dt \\ = (-1)^{n-1} n \int_{G/\varpi^{\mathbb{Z}}} \varphi_{\varpi}(g) \theta_{\pi}(g) dg = (-1)^{n-1} n \text{Tr}(\varphi_{\varpi}; \pi), \end{aligned}$$

as desired. ■

4 Faltings isomorphism

Here we freely use the notation in [Mie11, Section 2]. We need the following deep theorem due to Faltings ([Fal02], see also [FGL08] for more detailed exposition):

Theorem 4.1 *We have a $G \times D^{\times}$ -equivariant isomorphism $H_{\text{Dr}}^i \cong H_{\text{LT}}^i$ for every i .*

Note that the proof of Faltings' theorem does not require automorphic method. It gives the following very important finiteness result on H_{Dr}^i .

Corollary 4.2 *The G -representation H_{Dr}^i is admissible.*

Proof. Put $K_m = \text{Ker}(\text{GL}_n(\mathcal{O}) \rightarrow \text{GL}_n(\mathcal{O}/(\varpi^m)))$. Since

$$(H_{\text{LT}}^i)^{K_m} = H_c^i((M_m/\varpi^{\mathbb{Z}}) \otimes_{\bar{F}} \bar{F}, \bar{\mathbb{Q}}_{\ell})$$

is finite-dimensional, H_{LT}^i is an admissible representation of G . Thus, by Theorem 4.1, H_{Dr}^i is also an admissible representation of G . ■

Corollary 4.3 *For every $\pi \in \mathbf{Irr}(G/\varpi^{\mathbb{Z}})$ and integers $i, j \geq 0$, $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}}^i, \pi)$ is a finite-dimensional smooth representation of D^\times . Moreover, $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}}^i, \pi) = 0$ if $j \geq n$.*

Proof. Let \mathfrak{s} be the cuspidal support of π , and $H_{\mathrm{Dr}, \mathfrak{s}}^i$ be the \mathfrak{s} -component of H_{Dr}^i . Clearly we have $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}}^i, \pi) = \mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}, \mathfrak{s}}^i, \pi)$.

Let us observe that $H_{\mathrm{Dr}, \mathfrak{s}}^i$ is a finitely generated G -representation. Since $H_{\mathrm{Dr}, \mathfrak{s}}^i$ is an admissible G -representation by Corollary 4.2, it is $\mathfrak{Z}(G)$ -admissible, where $\mathfrak{Z}(G)$ denotes the Bernstein center of G (cf. [Ber84, §3.1]). Therefore, [Ber84, Corollaire 3.10] tells us that $H_{\mathrm{Dr}, \mathfrak{s}}^i$ is finitely generated. In particular, there exists a compact open subgroup $K' \subset D^\times$ which acts on $H_{\mathrm{Dr}, \mathfrak{s}}^i$ trivially.

Therefore, by [SS97, Corollary II.3.3], $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}, \mathfrak{s}}^i, \pi)$ is finite-dimensional and vanishes if $j \geq n$. The natural action of D^\times on $\mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}, \mathfrak{s}}^i, \pi)$ is smooth, since the action of K' is trivial. This completes the proof. \blacksquare

Definition 4.4 For $\pi \in \mathbf{Irr}(G/\varpi^{\mathbb{Z}})$, put

$$H_{\mathrm{Dr}}[\pi] = \sum_{i, j \geq 0} (-1)^{i+j} \mathrm{Ext}_{G/\varpi^{\mathbb{Z}}}^j(H_{\mathrm{Dr}}^i, \pi) \quad \text{in } R(D^\times/\varpi^{\mathbb{Z}}).$$

We can consider the character $\theta_{H_{\mathrm{Dr}}[\pi]}$ of $H_{\mathrm{Dr}}[\pi]$. Theorem 3.8 can be written in the following way:

Theorem 4.5 *For every $\pi \in \mathbf{Irr}(G/\varpi^{\mathbb{Z}})$ and $h \in (D^\times)^{\mathrm{reg}}$, we have*

$$\theta_{H_{\mathrm{Dr}}[\pi]}(h) = n\theta_\pi(g_h).$$

Proof. Theorem 3.8 says that $\mathrm{Tr}(\varphi^D; H_{\mathrm{Dr}}[\pi]) = (-1)^{n-1}n \mathrm{Tr}(\varphi; \pi)$. We can use exactly the same method as in the proof of [Mie11, Theorem 4.3]. \blacksquare

The following is another consequence of Theorem 4.1:

Corollary 4.6 *For every $\rho \in \mathbf{Irr}(D^\times/\varpi^{\mathbb{Z}})$, $H_{\mathrm{Dr}}^i[\rho] = \mathrm{Hom}_{D^\times}(H_{\mathrm{Dr}}^i, \rho)^{\mathrm{sm}}$ is a smooth G -representation of finite length. Moreover $H_{\mathrm{Dr}}^i[\rho]$ is isomorphic to $\mathrm{Hom}_{D^\times}(\rho, H_{\mathrm{Dr}}^i)^\vee$.*

Proof. By Proposition 2.1, Corollary 4.2 and [Mie11, Lemma 5.2], $H_{\mathrm{Dr}}^i[\rho]$ is a smooth G -representation of finite length. In the proof of [Mie11, Lemma 5.2], a G -equivariant injection $H_{\mathrm{Dr}}^i[\rho] \hookrightarrow \mathrm{Hom}_{D^\times}(\rho, H_{\mathrm{Dr}}^i)^\vee$ is constructed. It is easy to see that it is actually an isomorphism. \blacksquare

Definition 4.7 For $\rho \in \mathbf{Irr}(D^\times/\varpi^{\mathbb{Z}})$, put

$$H_{\mathrm{Dr}}[\rho] = \sum_i (-1)^i H_{\mathrm{Dr}}^i[\rho] \quad \text{in } R(G/\varpi^{\mathbb{Z}}).$$

We can consider the character $\theta_{H_{\text{Dr}}[\rho]}$ of $H_{\text{Dr}}[\rho]$. Corollary 2.5 can be written in the following way:

Theorem 4.8 *For every $\rho \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$ and $h \in (D^\times)^{\text{reg}}$, we have*

$$\theta_{H_{\text{Dr}}[\rho]}(g_h) = n\theta_\rho(h).$$

Proof. We denote the natural homomorphism $R(G/\varpi^\mathbb{Z}) \longrightarrow K(G/\varpi^\mathbb{Z})$ by EP. By [Dat07, Lemma 3.7], the composite of

$$R(G/\varpi^\mathbb{Z}) \xrightarrow{\text{EP}} K(G/\varpi^\mathbb{Z}) \xrightarrow{\text{Rk}} \overline{\mathcal{H}}(G/\varpi^\mathbb{Z}) \xrightarrow{\vee} C^\infty(G^{\text{ell}})$$

coincides with $\pi \longmapsto \theta_\pi|_{G^{\text{ell}}}$ (it was originally proved in [SS97, Theorem III.4.23]). Therefore, by Corollary 2.5 and Corollary 4.6 we have

$$\theta_{H_{\text{Dr}}[\rho]}(g) = ((\vee \circ \text{Rk} \circ \text{EP})(H_{\text{Dr}}[\rho]))(g) = n\theta_\rho(h_g)$$

for every $g \in G^{\text{ell}}$. Hence $\theta_{H_{\text{Dr}}[\rho]}(g_h) = n\theta_\rho(h)$ for every $h \in (D^\times)^{\text{reg}}$, as desired. \blacksquare

Remark 4.9 The proof of [SS97, Theorem III.4.23] seems to use [Kaz86, Theorem 0], whose proof relies on global technique. However, Theorem 4.1 and [Mie11, Theorem 4.3] give an alternative proof of Theorem 4.8, which does not involve any global argument.

5 Complements on representation theory

For locally constant class functions φ_1, φ_2 on $G^{\text{ell}}/\varpi^\mathbb{Z}$, put

$$\langle \varphi_1, \varphi_2 \rangle_{\text{ell}} = \sum_T \frac{1}{\#W_T} \int_{T/\varpi^\mathbb{Z}} D(t) \varphi_1(t) \overline{\varphi_2(t)} dt,$$

where T runs through conjugacy classes of elliptic maximal tori of G . Other notations are also the same as in the proof of Theorem 3.8.

For locally constant functions ϕ_1, ϕ_2 on $D^\times/\varpi^\mathbb{Z}$, put

$$\langle \phi_1, \phi_2 \rangle = \int_{D^\times/\varpi^\mathbb{Z}} \phi_1(h) \overline{\phi_2(h)} dh,$$

where the measure dh is normalized so that the volume of the compact group $D^\times/\varpi^\mathbb{Z}$ is one.

These two pairings are compatible, in the sense of the following lemma:

Lemma 5.1 *Let φ_1, φ_2 be locally constant class functions on $G^{\text{ell}}/\varpi^\mathbb{Z}$, and ϕ_1, ϕ_2 locally constant class functions on $D^\times/\varpi^\mathbb{Z}$. Assume that $\varphi_i(g_h) = \phi_i(h)$ for every $h \in (D^\times)^{\text{reg}}$. Then, we have*

$$\langle \varphi_1, \varphi_2 \rangle_{\text{ell}} = \langle \phi_1, \phi_2 \rangle.$$

Proof. Clear from Weyl's integral formula for D^\times . ■

The following orthogonality relation of characters is very important for our work.

Proposition 5.2 *For $\pi_1, \pi_2 \in \mathbf{Disc}(G/\varpi^\mathbb{Z})$, we have*

$$\langle \theta_{\pi_1}, \theta_{\pi_2} \rangle_{\text{ell}} = \begin{cases} 1 & \pi_1 \cong \pi_2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let ω_1, ω_2 be the central characters of π_1, π_2 , respectively. Then they are unitary, since $F^\times/\varpi^\mathbb{Z}$ is compact. If $\omega_1 = \omega_2$, then the lemma follows immediately from [Rog83, Lemma 5.3]. Otherwise,

$$\begin{aligned} \int_{T/\varpi^\mathbb{Z}} D(t) \theta_{\pi_1}(t) \overline{\theta_{\pi_2}(t)} dt &= \int_{T/Z_G} \left(\int_{Z_G/\varpi^\mathbb{Z}} D(tz) \theta_{\pi_1}(tz) \overline{\theta_{\pi_2}(tz)} dz \right) dt \\ &= \int_{T/Z_G} \left(\int_{Z_G/\varpi^\mathbb{Z}} \omega_1(z) \overline{\omega_2(z)} dz \right) D(t) \theta_{\pi_1}(t) \overline{\theta_{\pi_2}(t)} dt = 0, \end{aligned}$$

as desired. ■

Remark 5.3 The proof of [Rog83, Theorem 5.3] (for example, [DKV84, §A.3, §A.4]) seems to need a global argument, such as Howe's conjecture due to Clozel. However, at least if n is prime, we can give a purely local proof of Proposition 5.2 as follows. Here we use freely the notation which will be introduced in the next section.

Note that, if n is prime, then any irreducible discrete series representation of G is either a twisted Steinberg representation or supercuspidal (*cf.* [Zel80, Theorem 9.3]). First assume that both π_1 and π_2 are twisted Steinberg representations, and write $\pi_1 = \mathbf{St}_{\chi_1}$ and $\pi_2 = \mathbf{St}_{\chi_2}$, where χ_1 and χ_2 are characters of $F^\times/\varpi^\mathbb{Z}$. Then, by Lemma 5.1 and Lemma 6.7, we have

$$\langle \theta_{\mathbf{St}_{\chi_1}}, \theta_{\mathbf{St}_{\chi_2}} \rangle_{\text{ell}} = \langle \theta_{\chi_1 \circ \text{Nr}}, \theta_{\chi_2 \circ \text{Nr}} \rangle = \int_{F^\times/\varpi^\mathbb{Z}} \chi_1(z) \overline{\chi_2(z)} dz = \begin{cases} 1 & \chi_1 = \chi_2, \\ 0 & \chi_1 \neq \chi_2. \end{cases}$$

Since $\chi_1 \neq \chi_2$ implies $\mathbf{St}_{\chi_1} \not\cong \mathbf{St}_{\chi_2}$ (their characters are different), we have the orthogonality relation in this case.

Next assume that π_2 is supercuspidal. Then, by [DKV84, §A.3.e, §A.3.g], we can find a matrix coefficient $\phi \in \mathcal{H}(G/\varpi^\mathbb{Z})$ of π_2 satisfying the following:

- (a) $O_g^{G/\varpi^\mathbb{Z}}(\phi) = \overline{\theta_{\pi_2}(g)}$ for $g \in G^{\text{ell}}$ and $\int_{Z(g) \backslash G} \phi(x^{-1}gx) dx = 0$ for $g \in G^{\text{reg}} \setminus G^{\text{ell}}$.
- (b) $\text{Tr}(\phi; \pi_2) = 1$ and $\text{Tr}(\phi; \pi_1) = 0$ for $\pi_1 \in \mathbf{Disc}(G/\varpi^\mathbb{Z})$ with $\pi_1 \not\cong \pi_2$.

By (a) and Weyl's integral formula, we have

$$\begin{aligned} \text{Tr}(\phi; \pi_1) &= \int_{G/\varpi^\mathbb{Z}} \phi(g) \theta_{\pi_1}(g) dg = \sum_T \frac{1}{\#W_T} \int_{T/\varpi^\mathbb{Z}} D(t) O_t^{G/\varpi^\mathbb{Z}}(\phi) \theta_{\pi_1}(t) dt \\ &= \sum_T \frac{1}{\#W_T} \int_{T/\varpi^\mathbb{Z}} D(t) \overline{\theta_{\pi_2}(t)} \theta_{\pi_1}(t) dt = \langle \theta_{\pi_1}, \theta_{\pi_2} \rangle_{\text{ell}}. \end{aligned}$$

Therefore (b) gives the desired orthogonality relation.

6 Local Jacquet-Langlands correspondence

Let $R_I(G)$ be the submodule of $R(G)$ generated by the image of parabolically induced representations (cf. [Kaz86]) and put $\overline{R}(G) = R(G)/R_I(G)$. It is known that $\overline{R}(G)$ is a free \mathbb{Z} -module with a basis $\{[\pi] \mid \pi \in \mathbf{Disc}(G)\}$ (cf. [Dat07, Lemme 2.1.4]). We regard $\mathbf{Disc}(G)$ as a subset of $\overline{R}(G)$. Recall that the character of a parabolically induced representation vanishes on G^{ell} . Therefore, $\pi \mapsto \theta_\pi|_{G^{\text{ell}}}$ induces a map $\overline{R}(G) \longrightarrow C^\infty(G^{\text{ell}})$.

Moreover, we denote by $\overline{R}(G/\varpi^\mathbb{Z})$ the image of $R(G/\varpi^\mathbb{Z})$ in $\overline{R}(G)$. It is easy to see that $\{[\pi] \mid \pi \in \mathbf{Disc}(G/\varpi^\mathbb{Z})\}$ gives a basis of $\overline{R}(G/\varpi^\mathbb{Z})$. Set $\overline{R}(G/\varpi^\mathbb{Z})_\mathbb{Q} = \overline{R}(G/\varpi^\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Q}$ and $\widetilde{R}(G) = \overline{R}(G) \otimes_\mathbb{Z} \mathbb{Q}$.

Lemma 6.1 *The homomorphism $\widetilde{LJ}: R(G/\varpi^\mathbb{Z}) \longrightarrow R(D^\times/\varpi^\mathbb{Z})_\mathbb{Q}$ given by*

$$\pi \longmapsto \frac{(-1)^{n-1}}{n} H_{\text{Dr}}[\pi] \quad \text{for } \pi \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$$

factors through $\overline{R}(G/\varpi^\mathbb{Z})$.

Proof. Let P be a proper parabolic subgroup of G with Levi factor M and σ an irreducible smooth representation of M on which $\varpi \in Z_M$ acts trivially. By Theorem 4.5, the character of $\widetilde{LJ}(\text{Ind}_P^G \sigma)$ on $(D^\times)^{\text{reg}}$ is given by $h \mapsto (-1)^{n-1} \theta_{\text{Ind}_P^G \sigma}(g_h) = 0$. Since $(D^\times)^{\text{reg}}$ is dense in D^\times , it vanishes for every $h \in D^\times$. Thus $\widetilde{LJ}(\text{Ind}_P^G \sigma) = 0$ by linear independence of characters. Since the kernel of $R(G/\varpi^\mathbb{Z}) \longrightarrow \overline{R}(G/\varpi^\mathbb{Z})$ is generated by such representations as $\text{Ind}_P^G \sigma$, we conclude the proof. \blacksquare

The following is the main construction in this paper.

Definition 6.2 We define two homomorphisms

$$JL: R(D^\times/\varpi^\mathbb{Z})_\mathbb{Q} \longrightarrow \overline{R}(G/\varpi^\mathbb{Z})_\mathbb{Q}, \quad LJ: \overline{R}(G/\varpi^\mathbb{Z})_\mathbb{Q} \longrightarrow R(D^\times/\varpi^\mathbb{Z})_\mathbb{Q}$$

by

$$JL(\rho) = \frac{(-1)^{n-1}}{n} H_{\text{Dr}}[\rho], \quad LJ(\pi) = \frac{(-1)^{n-1}}{n} H_{\text{Dr}}[\pi]$$

for $\rho \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$ and $\pi \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$. The latter map is well-defined by the previous lemma.

Proposition 6.3 i) *We have the character relations*

$$\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h), \quad \theta_\pi(g_h) = (-1)^{n-1} \theta_{LJ(\pi)}(h)$$

for every $\rho \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$, $\pi \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$ and $h \in (D^\times)^{\text{reg}}$.

ii) *For every $\rho, \rho' \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$ and $\pi, \pi' \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$, we have*

$$\begin{aligned} \langle \theta_{JL(\rho)}, \theta_{JL(\rho')} \rangle_{\text{ell}} &= \langle \theta_\rho, \theta_{\rho'} \rangle, & \langle \theta_{LJ(\pi)}, \theta_{LJ(\pi')} \rangle &= \langle \theta_\pi, \theta_{\pi'} \rangle_{\text{ell}}, \\ \langle \theta_{JL(\rho)}, \theta_\pi \rangle_{\text{ell}} &= \langle \theta_\rho, \theta_{LJ(\pi)} \rangle. \end{aligned}$$

- iii) Two maps JL and LJ are inverse to each other.
- iv) The map JL is compatible with character twists. Namely, for a character χ of F^\times which is trivial on $\varpi^{n\mathbb{Z}} \subset F^\times$, we have $JL(\rho \otimes (\chi \circ \text{Nrd})) = JL(\rho) \otimes (\chi \circ \det)$. The same holds for LJ .
- v) The map JL preserves central characters. Namely, for $\rho \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$, write $JL(\rho) = \sum_{\pi \in \mathbf{Disc}(G/\varpi^\mathbb{Z})} a_\pi[\pi]$. Then, every π with $a_\pi \neq 0$ has the same central character as ρ . The same holds for LJ .

Proof. i) is clear from Theorem 4.5 and Theorem 4.8. ii) follows from i) and Lemma 5.1.

Prove iii). For $\pi \in \mathbf{Disc}(G/\varpi^\mathbb{Z})$, write $JL(LJ(\pi)) = \sum_{\pi' \in \mathbf{Disc}(G/\varpi^\mathbb{Z})} a_{\pi'}[\pi']$. Then, by ii) and Proposition 5.2 we have

$$a_{\pi'} = \langle \theta_{JL(LJ(\pi))}, \theta_{\pi'} \rangle_{\text{ell}} = \langle \theta_{LJ(\pi)}, \theta_{LJ(\pi')} \rangle = \langle \theta_\pi, \theta_{\pi'} \rangle_{\text{ell}}.$$

Therefore, $a_{\pi'} = 1$ if $\pi' = \pi$, and $a_{\pi'} = 0$ otherwise. In other words, $JL(LJ(\pi)) = \pi$. Thus we have $JL \circ LJ = \text{id}$. Similarly we can prove that $LJ \circ JL = \text{id}$.

For iv), it suffices to show that LJ is compatible with character twists. Let $\pi \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$ and χ be a character of F^\times which is trivial on $\varpi^{n\mathbb{Z}} \subset F^\times$. Then, for every $h \in (D^\times)^{\text{reg}}$, we have

$$\begin{aligned} \theta_{LJ(\pi \otimes (\chi \circ \det))}(h) &= (-1)^{n-1} \theta_{\pi \otimes (\chi \circ \det)}(g_h) = (-1)^{n-1} \chi(\det g_h) \theta_\pi(g_h) \\ &= \chi(\text{Nrd } h) \theta_{LJ(\pi)}(h) = \theta_{LJ(\pi) \otimes (\chi \circ \text{Nrd})}(h). \end{aligned}$$

Since $(D^\times)^{\text{reg}}$ is dense in D^\times , we have $\theta_{LJ(\pi \otimes (\chi \circ \det))} = \theta_{LJ(\pi) \otimes (\chi \circ \text{Nrd})}$. By linear independence of characters, we conclude that $LJ(\pi \otimes (\chi \circ \det)) = LJ(\pi) \otimes (\chi \circ \text{Nrd})$.

Finally we prove v). Write $JL(\rho) = \sum_{\pi \in \mathbf{Disc}(G/\varpi^\mathbb{Z})} a_\pi[\pi]$. By Proposition 5.2, $a_\pi = \langle \theta_{JL(\rho)}, \theta_\pi \rangle_{\text{ell}}$. Assume that the central character of $\pi \in \mathbf{Disc}(G/\varpi^\mathbb{Z})$ is different from that of ρ . Then, we have

$$a_\pi = \langle \theta_{JL(\rho)}, \theta_\pi \rangle_{\text{ell}} = \sum_T \frac{1}{\#W_T} \int_{T/\varpi^\mathbb{Z}} D(t) \theta_{JL(\rho)}(t) \overline{\theta_\pi(t)} dt = 0$$

in the same way as in the proof of Proposition 5.2. Therefore JL preserves central characters. By this result and ii), we have $\langle \theta_\rho, \theta_{LJ(\pi)} \rangle = 0$ unless ρ and π have the same central character. This means that the coefficient of $\rho \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$ in $LJ(\pi)$ is zero unless the central character of ρ is the same as that of π . Namely, LJ preserves central characters. \blacksquare

By twisting, we can extend JL and LJ to maps between $R(D^\times)$ and $\overline{R}(G)$.

Proposition 6.4 *There exists a unique extension of JL to a homomorphism from $R(D^\times)_\mathbb{Q}$ to $\overline{R}(G)_\mathbb{Q}$ which is compatible with character twists. Similarly, we have a unique extension of LJ to a homomorphism $\overline{R}(G)_\mathbb{Q} \longrightarrow R(D^\times)_\mathbb{Q}$ which is compatible with character twists. We denote them by JL and LJ again. These are inverse to each other, satisfy the same character relations as in Proposition 6.3 i), and preserve central characters.*

Proof. For $\rho \in \mathbf{Irr}(D^\times)$, let ω_ρ be its central character. Take $c \in \mathbb{C}^\times$ such that $c^n = \omega_\rho(\varpi)$, and consider the character $\chi_c: z \mapsto c^{v_F(z)}$ of F^\times . Then $\rho \otimes (\chi_c^{-1} \circ \text{Nrd}) \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$. Extend JL to $R(D^\times)_\mathbb{Q} \longrightarrow \overline{R}(G)_\mathbb{Q}$ by

$$JL(\rho) = JL(\rho \otimes (\chi_c^{-1} \circ \text{Nrd})) \otimes (\chi_c \circ \det).$$

By Proposition 6.3 iv), it is independent of the choice of c . Moreover, we can easily observe that it is the unique extension of the original JL which is compatible with character twists. Similarly, we can uniquely extend LJ to a map $\overline{R}(G)_\mathbb{Q} \longrightarrow R(D^\times)_\mathbb{Q}$ compatible with character twists.

By using Proposition 6.3 v), we can easily check that the extended JL and LJ are inverse to each other. The remaining parts are also immediate consequences of Proposition 6.3 i), v). \blacksquare

Next we will observe the uniqueness of the maps JL, LJ satisfying the character relations.

Proposition 6.5 i) Let $JL': R(D^\times)_\mathbb{Q} \longrightarrow \overline{R}(G)_\mathbb{Q}$ be a homomorphism satisfying the character relation $\theta_\rho(h) = (-1)^{n-1} \theta_{JL'(\rho)}(g_h)$ for every $h \in (D^\times)^{\text{reg}}$. Then we have $JL' = JL$.

ii) Let $LJ': \overline{R}(G)_\mathbb{Q} \longrightarrow R(D^\times)_\mathbb{Q}$ be a homomorphism satisfying the character relation $\theta_\pi(g_h) = (-1)^{n-1} \theta_{LJ'(\pi)}(h)$ for every $h \in (D^\times)^{\text{reg}}$. Then we have $LJ' = LJ$.

Proof. To prove i), it suffices to show that $LJ \circ JL' = \text{id}$. By the character relation, we have $\theta_{LJ(JL'(\rho))}(h) = \theta_\rho(h)$ for every $\rho \in \mathbf{Irr}(D^\times)$ and $h \in (D^\times)^{\text{reg}}$. Thus we can conclude that $LJ(JL'(\rho)) = \rho$ by linear independence of characters for D^\times . For ii), prove $LJ' \circ LJ = \text{id}$ by a similar argument. \blacksquare

So far, we have obtained the following theorem:

Theorem 6.6 We can construct the following two homomorphisms geometrically:

$$JL: R(D^\times)_\mathbb{Q} \longrightarrow \overline{R}(G)_\mathbb{Q}, \quad LJ: \overline{R}(G)_\mathbb{Q} \longrightarrow R(D^\times)_\mathbb{Q}.$$

These two maps are inverse to each other, and satisfy the character relations

$$\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h), \quad \theta_\pi(g_h) = (-1)^{n-1} \theta_{LJ(\pi)}(h)$$

for every $h \in (D^\times)^{\text{reg}}$. They are characterized by these character relations. Moreover, JL and LJ are compatible with character twists, and preserve central characters.

Let $B \subset G$ be the Borel subgroup consisting of upper triangular matrices. Recall that the Steinberg representation \mathbf{St} is the unique irreducible quotient of the unnormalized induction $\text{Ind}_B^G \mathbf{1}$ from the trivial character $\mathbf{1}$ on B . For a character χ of F^\times , put $\mathbf{St}_\chi = \mathbf{St} \otimes (\chi \circ \det)$. A representation of the form \mathbf{St}_χ is called a twisted Steinberg representation. It is an irreducible discrete series representation of G . The following lemma is very well-known:

Lemma 6.7 *We have $\theta_{\chi \circ \text{Nrd}}(h) = (-1)^{n-1} \theta_{\mathbf{St}_\chi}(g_h)$ for a character χ of F^\times and $h \in (D^\times)^{\text{reg}}$.*

Proof. In $\overline{R}(G)$, we have $[\mathbf{St}_\chi] = (-1)^{n-1} [\chi \circ \det]$ (cf. [Dat07, Remarque 2.1.14]). As the character of a parabolically induced representation vanishes on G^{ell} , we have

$$\begin{aligned} \theta_{\mathbf{St}_\chi}(g_h) &= (-1)^{n-1} \theta_{\chi \circ \det}(g_h) = (-1)^{n-1} \chi(\det g_h) = (-1)^{n-1} \chi(\text{Nrd } h) \\ &= (-1)^{n-1} \theta_{\chi \circ \text{Nrd}}(h), \end{aligned}$$

as desired. ■

Corollary 6.8 *For a character χ of F^\times , we have $JL(\chi \circ \text{Nrd}) = \mathbf{St}_\chi$ and $LJ(\mathbf{St}_\chi) = \chi \circ \text{Nrd}$.*

Proof. By Lemma 6.7, we have $\theta_{LJ(\mathbf{St}_\chi)} = \theta_{\chi \circ \text{Nrd}}$. Linear independence of characters tells us that $LJ(\mathbf{St}_\chi) = \chi \circ \text{Nrd}$. ■

The following is a consequence of the non-cuspidality result in [Mie10b]:

Proposition 6.9 *For an irreducible supercuspidal representation π of G , write $LJ(\pi) = \sum_{\rho \in \mathbf{Irr}(D^\times)} a_\rho [\rho]$. Then we have $a_\rho \geq 0$ for every ρ .*

Proof. We may assume that $\pi \in \mathbf{Irr}(G/\varpi^\mathbb{Z})$. Since π is injective in the category of smooth $G/\varpi^\mathbb{Z}$ -representations, $\text{Ext}_{G/\varpi^\mathbb{Z}}^j(H_{\text{Dr}}^i, \pi) = 0$ unless $j = 0$. By Theorem 4.1 and [Mie10b, Theorem 3.7], we have $\text{Hom}_{G/\varpi^\mathbb{Z}}(H_{\text{Dr}}^i, \pi) = 0$ unless $i = n - 1$. Therefore we have $LJ(\pi) = n^{-1} [\text{Hom}_{G/\varpi^\mathbb{Z}}(H_{\text{Dr}}^{n-1}, \pi)]$. This concludes the proof. ■

Now we can prove the local Jacquet-Langlands correspondence for prime n .

Theorem 6.10 *Assume that n is a prime number. Then JL induces a bijection*

$$JL: \mathbf{Irr}(D^\times) \xrightarrow{\cong} \mathbf{Disc}(G)$$

satisfying the character relation $\theta_\rho(h) = (-1)^{n-1} \theta_{JL(\rho)}(g_h)$ for every $h \in (D^\times)^{\text{reg}}$.

Proof. For simplicity, we denote by $\mathbf{Cusp}(G/\varpi^\mathbb{Z})$ the subset of $\mathbf{Disc}(G/\varpi^\mathbb{Z})$ consisting of supercuspidal representations. By Theorem 6.6, it suffices to show the following:

- (a) For $\rho \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$, $JL(\rho) \in \mathbf{Disc}(G/\varpi^\mathbb{Z})$.
- (b) For $\pi \in \mathbf{Disc}(G/\varpi^\mathbb{Z})$, $LJ(\pi) \in \mathbf{Irr}(D^\times/\varpi^\mathbb{Z})$.

First we shall prove (a). If ρ is a character, then it follows from Corollary 6.8. Assume that ρ is not a character, and write $JL(\rho) = \sum_{\pi \in \mathbf{Disc}(G/\varpi^{\mathbb{Z}})} a_{\pi}[\pi]$. Then, $a_{\pi} = 0$ unless π is supercuspidal. Indeed, if $\pi \notin \mathbf{Cusp}(G/\varpi^{\mathbb{Z}})$, then π is a twisted Steinberg representation \mathbf{St}_{χ} , for n is prime (cf. [Zel80, Theorem 9.3]). By Proposition 5.2, Proposition 6.3 ii) and Corollary 6.8, we have

$$a_{\pi} = a_{\mathbf{St}_{\chi}} = \langle \theta_{JL(\rho)}, \theta_{\mathbf{St}_{\chi}} \rangle_{\text{ell}} = \langle \theta_{\rho}, \theta_{LJ(\mathbf{St}_{\chi})} \rangle = \langle \theta_{\rho}, \theta_{\chi \circ \text{Nrd}} \rangle = 0.$$

Since $JL(\rho) \neq 0$, there exists at least one $\pi \in \mathbf{Cusp}(G/\varpi^{\mathbb{Z}})$ satisfying $a_{\pi} \neq 0$. Let us observe that such π is unique. Assume that there exist $\pi, \pi' \in \mathbf{Cusp}(G/\varpi^{\mathbb{Z}})$ such that a_{π} and $a_{\pi'}$ are non-zero. Then, we have $\langle \theta_{LJ(\pi)}, \theta_{\rho} \rangle = \langle \theta_{\pi}, \theta_{JL(\rho)} \rangle_{\text{ell}} = a_{\pi} \neq 0$, and similarly $\langle \theta_{LJ(\pi')}, \theta_{\rho} \rangle \neq 0$. In other words, if we write

$$LJ(\pi) = \sum_{\varrho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})} b_{\varrho}[\varrho], \quad LJ(\pi') = \sum_{\varrho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})} b'_{\varrho}[\varrho],$$

then $b_{\rho} = a_{\pi}$ and $b'_{\rho} = a_{\pi'}$ are non-zero. On the other hand, Proposition 6.9 tells us that $b_{\varrho} \geq 0$ and $b'_{\varrho} \geq 0$ for every ϱ . Thus, by Proposition 6.3 ii), we conclude that

$$\langle \theta_{\pi}, \theta_{\pi'} \rangle_{\text{ell}} = \langle \theta_{LJ(\pi)}, \theta_{LJ(\pi')} \rangle = \sum_{\varrho} b_{\varrho} b'_{\varrho} > 0,$$

which is equivalent to $\pi \cong \pi'$ by Proposition 5.2. Now we have $JL(\rho) = a_{\pi}[\pi]$ for some $\pi \in \mathbf{Cusp}(G/\varpi^{\mathbb{Z}})$. Moreover, the argument above tells us that $a_{\pi} \geq 0$. By Proposition 5.2 and Proposition 6.3 ii), we have

$$1 = \langle \theta_{\rho}, \theta_{\rho} \rangle = \langle \theta_{JL(\rho)}, \theta_{JL(\rho)} \rangle_{\text{ell}} = a_{\pi}^2 \langle \theta_{\pi}, \theta_{\pi} \rangle_{\text{ell}} = a_{\pi}^2.$$

Hence we conclude that $a_{\pi} = 1$ and $JL(\rho) = \pi \in \mathbf{Disc}(G/\varpi^{\mathbb{Z}})$.

Next prove (b). Write $LJ(\pi) = \sum_{\rho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})} b_{\rho}[\rho]$. Then, since JL and LJ are inverse to each other, we have $\pi = \sum_{\rho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})} b_{\rho} JL(\rho)$, and thus

$$1 = \sum_{\rho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})} b_{\rho} \langle \theta_{JL(\rho)}, \theta_{\pi} \rangle_{\text{ell}}.$$

By (a) and Proposition 5.2, there exists $\rho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})$ such that $\pi = JL(\rho)$. Then $LJ(\pi) = \rho \in \mathbf{Irr}(D^{\times}/\varpi^{\mathbb{Z}})$, as desired. ■

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